XXIV. Bivariate Density and Distribution Functions

Bivariate Density Functions

- We have two variables, for example:

$$\begin{aligned} Y_1 &:= \text{\# of math units} \\ Y_2 &:= \text{\# of CS units} \end{aligned}$$

- We have a bivariate density function

$$f(y_1, y_2) := \text{students with } y_1 \text{ math units and } y_2 \text{ CS units}$$

These are the numbers of units that any given student has taken. f gives the density of students who have taken a particular combination of numbers of units.

Properties of the Density Function

1. $f(y_1, y_2) \geq 0$.
2. $\displaystyle\int_{y_1=-\infty}^{y_1=\infty} \int_{y_2=-\infty}^{y_2=\infty} f(y_1, y_2)\, dy_2\, dy_1 = 1$.

- We can calculate probabilities:

$$\begin{aligned} &P(a \leq Y_1 \leq b, c \leq Y_2 \leq d) \\ &= \int_{y_1=a}^{y_1=b} \int_{y_2=c}^{y_2=d} f(y_1, y_2)\, dy_2\, dy_1 \end{aligned}$$

- If discrete, change f to p, $\int$ to $\sum$.

You can't have a negative number of kids with any particular combination, and the total density of kids is 1. [Graph.]

Bivariate Distribution Functions

- The bivariate distribution function is

$$\begin{aligned} F(y_1, y_2) &:= P(Y_1 \le y_1, Y_2 \le y_2) \\ &= \int_{t_1=-\infty}^{t_1=y_1} \int_{t_2=-\infty}^{t_2=y_2} f(t_1, t_2)\, dt_2\, dt_1. \end{aligned}$$

- Properties:
  1. $F(y_1, -\infty) = F(-\infty, y_2) = 0$, for all y_1, y_2.
  2. $F(\infty, \infty) = 1$.

Example I

Consider the joint density function $f(y_1, y_2) := ky_2$ on the triangle with corners at $(0, 0)$, $(0, 1)$, and $(1, 1)$. Find the value of k.

$$\begin{aligned}
\int_{y_2=0}^{y_2=1} \int_{y_1=0}^{y_1=y_2} k y_2 \, dy_1 \, dy_2 &= k \int_{y_2=0}^{y_2=1} y_1 y_2 \Big|_{y_1=0}^{y_1=y_2} \, dy_2 \\
&= k \int_{y_2=0}^{y_2=1} y_2^2 \, dy_2 \\
&= \frac{k}{3} \, y_2^3 \Big|_{y_2=0}^{y_2=1} = \frac{k}{3} = 1 \\
k &= \boxed{3}
\end{aligned}$$

Example II

As in Example I, consider the joint density function $f(y_1, y_2) := 3y_2$ on the triangle with corners at $(0,0)$, $(0,1)$, and $(1,1)$. Find $F\left(\frac{1}{3}, \frac{1}{2}\right)$.

(Graph.)

$$\begin{aligned}
F\left(\frac{1}{3},\frac{1}{2}\right) &:= P\left(Y_1<\frac{1}{3},Y_2<\frac{1}{2}\right)\\
&= \int_{y_1=0}^{y_1=\frac{1}{3}}\int_{y_2=y_1}^{y_2=\frac{1}{2}} 3y_2\,dy_2\,dy_1\\
&= \frac{3}{2}\int_{y_1=0}^{y_1=\frac{1}{3}} y_2^2\Big|_{y_2=y_1}^{y_2=\frac{1}{2}}\,dy_1\\
&= \frac{3}{2}\int_{y_1=0}^{y_1=\frac{1}{3}}\left(\frac{1}{4}-y_1^2\right)dy_1\\
&= \frac{3}{2}\left(\frac{1}{4}y_1-\frac{1}{3}y_1^3\right)\Big|_{y_1=0}^{y_1=\frac{1}{3}}\\
&= \frac{3}{2}\left(\frac{1}{12}-\frac{1}{81}\right)\\
&= \frac{3}{2}\left(\frac{1}{8}-\frac{1}{54}\right)\\
&= \frac{27-4}{216}=\boxed{\frac{23}{216}\approx 0.1064=10.64\%}
\end{aligned}$$

Example III

As in Example I, consider the joint density function $f(y_1,y_2) := 3y_2$ on the triangle with corners at $(0,0)$, $(0,1)$, and $(1,1)$. Find $P(2Y_1<Y_2)$.

(Graph.)

$$\begin{aligned}
P(2Y_1 < Y_2) &= \int_{y_2=0}^{y_2=1} \int_{y_1=0}^{y_1=\frac{y_2}{2}} 3y_2 \, dy_1 \, dy_2 \\
&= 3 \int_{y_2=0}^{y_2=1} y_1 y_2 \Big|_{y_1=0}^{y_1=\frac{y_2}{2}} \, dy_2 \\
&= \frac{3}{2} \int_{y_2=0}^{y_2=1} y_2^2 \, dy_2 \\
&= \frac{1}{2} \, y_2^3 \Big|_{y_2=0}^{y_2=1} = \boxed{\frac{1}{2}}
\end{aligned}$$

Example IV

Consider the joint density function

$$f(y_1, y_2) := e^{-(y_1+y_2)}$$

on the region $y_1 > 0$, $y_2 > 0$. Find $P(Y_1 \le 2, Y_2 \ge 3)$.

(Graph.)

$$
\begin{aligned}
P(Y_1 \leq 2, Y_2 \geq 3) &= \int_{y_2=3}^{y_2=\infty} \int_{y_1=0}^{y_1=2} e^{-(y_1+y_2)}\, dy_1\, dy_2 \\
&= \int_{y_2=3}^{y_2=\infty} e^{-y_2} \int_{y_1=0}^{y_1=2} e^{-y_1}\, dy_1\, dy_2 \\
&= \int_{y_2=3}^{y_2=\infty} e^{-y_2} \left(-e^{-y_1}\right)\Big|_{y_1=0}^{y_1=2}\, dy_2 \\
&= \left(1 - e^{-}2\right) \int_{y_2=3}^{y_2=\infty} e^{-y_2}\, dy_2 \\
&= \left(1 - e^{-}2\right) \left(-e^{-y_2}\right)\Big|_{y_2=3}^{y_2=\infty} \\
&= \left(1 - e^{-}2\right) \left(e^{-3}\right) \\
&= \boxed{e^{-3} - e^{-5} \approx 0.043 = 4.3\%}
\end{aligned}
$$

Example V

Consider the joint density function

$$f(y_1, y_2) := e^{-(y_1+y_2)}$$

on the region $y_1 > 0$, $y_2 > 0$. Find $P(Y_1 + Y_2 \leq 2)$.

(Graph.)

$$\begin{aligned}
P\left(Y_1+Y_2 \leq 2\right) &= \int_{y_1=0}^{y_1=2} \int_{y_2=0}^{y_2=2-y_1} e^{-(y_1+y_2)}\, dy_2\, dy_1 \\
&= \int_{y_1=0}^{y_1=2} e^{-y_1} \int_{y_2=0}^{y_2=2-y_1} e^{-y_2}\, dy_2\, dy_1 \\
&= \int_{y_1=0}^{y_1=2} e^{-y_1} \left(-e^{-y_2}\right)\Big|_{y_2=0}^{y_2=2-y_1} dy_1 \\
&= \int_{y_1=0}^{y_1=2} e^{-y_1}\left(-e^{y_1-2}+1\right) dy_1 \\
&= \int_{y_1=0}^{y_1=2} \left(-e^{-2}+e^{-y_1}\right) dy_1 \\
&= \left(-e^{-2} y_1 - e^{-y_1}\right)\Big|_{y_1=0}^{y_1=2} \\
&= -2e^{-2} - e^{-2} + 0 + 1 = \boxed{1-3e^{-2} \approx 0.594 = 59.4\%}
\end{aligned}$$