XXVIII. Expected Value of a Function of Random Variables

Review of Single Variable Case

- Expected value of a single variable:

$$E(Y) := \begin{cases} \displaystyle\sum_y yp(y) & \text{(discrete)} \\ \displaystyle\int_y yf(y)\,dy & \text{(continuous)} \end{cases}$$

- Expected value of a function $g(Y)$ (say, $g(Y) := Y^2$):

$$E\left[g(Y)\right] := \begin{cases} \displaystyle\sum_y g(y)p(y) & \text{(discrete)} \\ \displaystyle\int_y g(y)f(y)\,dy & \text{(continuous)} \end{cases}$$

Bivariate Case

- Expected value of a function $g\left(Y_1, Y_2\right)$:

$$E\left[g\left(Y_1, Y_2\right)\right] := \begin{cases} \displaystyle\sum\sum g\left(y_1, y_2\right) p\left(y_1, y_2\right) \\ \displaystyle\iint g\left(y_1, y_2\right) f\left(y_1, y_2\right) dy_2\, dy_1 \end{cases}$$

Linearity of Expectation

1. $E(c) = c$

2. $E\left[cg\left(Y_1, Y_2\right)\right] = cE\left[g\left(Y_1, Y_2\right)\right]$

3. Additivity:

$$\begin{aligned} & E\left[g_1\left(Y_1, Y_2\right) + g_2\left(Y_1, Y_2\right)\right] \\ = \; & E\left[g_1\left(Y_1, Y_2\right)\right] + E\left[g_2\left(Y_1, Y_2\right)\right] \end{aligned}$$

Example I

Let $f\left(y_1, y_2\right) := e^{-y_2}, 0 \leq y_1 \leq y_2 < \infty$. Calculate $E\left(Y_1 + Y_2\right)$.

(Graph.)

$$\begin{aligned} E\left(Y_1 + Y_2\right) & := \int_{y_2=0}^{y_2=\infty} \int_{y_1=0}^{y_1=y_2} \left(y_1 + y_2\right) e^{-y_2} \, dy_1 \, dy_2 \\ & = \int_{y_2=0}^{y_2=\infty} e^{-y_2} \int_{y_1=0}^{y_1=y_2} \left(y_1 + y_2\right) \, dy_1 \, dy_2 \\ & = \int_{y_2=0}^{y_2=\infty} e^{-y_2} \left(y_1 y_2 + \frac{y_2^2}{2}\right)\Bigg|_{y_1=0}^{y_1=y_2} dy_2 \\ & = \int_{y_2=0}^{y_2=\infty} \frac{3}{2} y_2^2 e^{-y_2} \, dy_2 \quad \text{Use tabular integration.} \\ & = \frac{3}{2}\left(-y_2^2 e^{-y_2} - 2y_2 e^{-y_2} - 2e^{-y_2}\right)\Big|_{y_2=0}^{y_2=\infty} = \boxed{3} \end{aligned}$$

Example II

Let $f\left(y_1, y_2\right) := 2\left(1 - y_2\right), 0 \leq y_1, y_2 \leq 1$. Calculate $E\left(Y_1 Y_2\right)$.

(Graph.)

$$\begin{aligned}
E(Y_1Y_2) &:= \int_{y_2=0}^{y_2=1}\int_{y_1=0}^{y_1=1} y_1y_2 2(1-y_2)\,dy_1\,dy_2 \\
&= \int_{y_2=0}^{y_2=1} y_1^2y_2(1-y_2)\Big|_{y_1=0}^{y_1=1}\,dy_2 \\
&= \int_{y_2=0}^{y_2=1} y_2(1-y_2)\,dy_2 \\
&= \left(\frac{y_2^2}{2}-\frac{y_2^3}{3}\right)\Bigg|_{y_2=0}^{y_2=1} = \frac{1}{2}-\frac{1}{3} = \boxed{\frac{1}{6}}
\end{aligned}$$

Example III

Let Y_1 and Y_2 have means $\mu_1 = 7$ and $\mu_2 = 5$. Let $U_1 := Y_1 + 2Y_2$ and $U_2 := Y_1 - Y_2$. Calculate $E(U_1)$ and $E(U_2)$.

$$E(U_1) = 7 + 2\cdot 5 = \boxed{17}, E(U_2) = 7 - 5 = \boxed{2}$$

Example IV

Let $f(y_1, y_2) := 6(1-y_2), 0 \le y_1 \le y_2 < 1$. Calculate $E(Y_1)$ and $E(Y_2)$.

(Graph.)

$$\begin{aligned}
E(Y_1) &:= \int_{y_2=0}^{y_2=1} \int_{y_1=0}^{y_1=y_2} y_1 6(1-y_2)\, dy_1\, dy_2 \\
&= \int_{y_2=0}^{y_2=1} 3y_1^2 (1-y_2)\Big|_{y_1=0}^{y_1=y_2} dy_2 \\
&= \int_{y_2=0}^{y_2=1} 3y_2^2 (1-y_2)\, dy_2 \\
&= \left(y_2^3 - \frac{3}{4} y_2^4 \right)\Bigg|_{y_2=0}^{y_2=1} = 1 - \frac{3}{4} = \boxed{\frac{1}{4}} \\
E(Y_2) &:= \int_{y_2=0}^{y_2=1} \int_{y_1=0}^{y_1=y_2} y_2 6(1-y_2)\, dy_1\, dy_2 \\
&= \int_{y_2=0}^{y_2=1} 6y_1 y_2 (1-y_2)\Big|_{y_1=0}^{y_1=y_2} dy_2 \\
&= \int_{y_2=0}^{y_2=1} 6y_2^2 (1-y_2)\, dy_2 \\
&= \left(2y_2^3 - \frac{3}{2} y_2^4 \right)\Bigg|_{y_2=0}^{y_2=1} = 2 - \frac{3}{2} = \boxed{\frac{1}{2}}
\end{aligned}$$

Example V

Let $f(y_1, y_2) := 6(1-y_2), 0 \le y_1 \le y_2 < 1$. Calculate $E(2Y_1 + 3Y_2)$.

$$E(2Y_1 + 3Y_2) = 2E(Y_1) + 3E(Y_2) = 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{2} = \boxed{2}$$